Name of College - S.S. College J-Bad
Deptt - Mathematics
Topic → Problem based On hyperbola (Analytical Geometry of 2D)
Class - B.Sc I (sub)
Date - 21-07-2020
Time - 1:00 P.M To 1:45 PM
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~~Problem~~ Study material :→

To Find the Equation of the chord Joining Two points $(a\sec\theta, b\tan\theta)$ and $(a\sec\varphi, b\tan\varphi)$ and then deduce the Equation of Tangent at the point $(a\sec\theta, b\tan\theta)$ to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

[Since the Equation of st. line passing through a given point (x_1, y_1) is

$$\frac{y-y_1}{x-x_1} = \left(\frac{dy}{dx}\right)_{x_1,y_1} = \text{slope of tangent at } (x_1,y_1),$$

and the Equation of a st. line passing through two points (x_1,y_1) and (x_2,y_2) is Given by

$$\frac{y-y_1}{x-x_1} = \frac{y_1-y_2}{x_1-x_2} = \quad]$$

Now Here Equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and Given points are $\left(\begin{array}{l} x_1 = a\sec\theta \\ y_1 = b\tan\theta \end{array}\right)$ and $\left(\begin{array}{l} x_2 = a\sec\varphi \\ y_2 = b\tan\varphi \end{array}\right)$

∴ Equation of chord of hyperbola is

$$\frac{y - b\tan\theta}{x - a\sec\theta} = \frac{b\tan\theta - b\tan\varphi}{a\sec\theta - a\sec\varphi}$$

[using Equation of st. line passing through (x_1,y_1) and (x_2,y_2)]

$$\Rightarrow \frac{y - b\tan\theta}{b(\tan\theta - \tan\varphi)} = \frac{x - a\sec\theta}{a(\sec\theta - \sec\varphi)}$$

$$\Rightarrow \frac{y - b\tan\theta}{b\left[\frac{\sin\theta}{\cos\theta} - \frac{\sin\phi}{\cos\phi}\right]} = \frac{x - a\sec\theta}{a\left[\frac{1}{\cos\theta} - \frac{1}{\cos\phi}\right]}$$

$$\Rightarrow \frac{(y - b\tan\theta)\cos\theta\cos\phi}{b\sin(\theta - \phi)} = \frac{(x - a\sec\theta)\cos\theta\cos\phi}{a(\cos\phi - \cos\theta)}$$

$$\Rightarrow \frac{y - b\tan\theta}{b\sin(\theta - \phi)} = \frac{x - a\sec\theta}{a \cdot 2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}}$$

$$\Rightarrow \frac{y - b\tan\theta}{b \times \cancel{2}\cancel{\sin\frac{\theta-\phi}{2}}\cos\frac{\theta-\phi}{2}} = \frac{x - a\sec\theta}{\cancel{2}a\sin\frac{\theta+\phi}{2}\cancel{\sin\frac{\theta-\phi}{2}}}$$

$$\Rightarrow \frac{y - \tan\theta}{b\cos\frac{\theta-\phi}{2}} = \frac{x - a\sec\theta}{a\sin\frac{\theta+\phi}{2}}$$

this is the required equation of chord joining two given points to Given Hyperbola

For Equation of Tangent -

For we must have $\phi \to \theta$

$$\Rightarrow \frac{y - b\tan\theta}{b\cos 0} = \frac{x - a\sec\theta}{a\sin\theta}$$

$\because \cos 0 = 1$

$$\Rightarrow \frac{y - b\tan\theta}{b} = \frac{x - a\sec\theta}{a\sin\theta}$$

$$\Rightarrow \frac{y}{b} - \tan\theta = \frac{x}{a\sin\theta} - \frac{1}{\sin\theta\cos\theta}$$

$$\Rightarrow \frac{x}{a\sin\theta} - y/b = \frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \frac{bx - ay\sin\theta}{ab\cancel{\sin\theta}} = \frac{1 - \sin^2\theta}{\cancel{\sin\theta}\cos\theta}$$

$$\Rightarrow \frac{bx - ay\sin\theta}{ab} = \cos\theta$$

$$\Rightarrow bx - ay\sin\theta = ab\cos\theta$$

this is the required equation of tangent at point θ to given hyperbola

To Find the Equation of the director circle of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Director circle :- The locus of point of intersection of two mutual perpendicular tangents to a hyperbola is a circle known as director circle

Since By the Question
Equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{— (I)}$$

$\therefore$ the Equation of any tangent to (I) is

$$y = mx + \sqrt{a^2m^2 - b^2} \quad \text{— (II)}$$

Let $P(x_1, y_1)$ be the point of intersection of two mutual perpendicular tangents
Since the tangent (II) passes through (x_1, y_1)
we have

$$y_1 = mx_1 + \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y_1 - mx_1 = \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow (y_1 - mx_1)^2 = a^2m^2 - b^2$$

$$\Rightarrow y_1^2 + m^2x_1^2 - 2mx_1y_1 = a^2m^2 - b^2$$

$$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 + b^2) = 0$$

As it is a quadratic Equation in m
and there for it has two roots m_1 and m_2

then $$m_1 m_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2}$$

Now corresponding to the values m_1 and m_2 we get two tangents to the hyperbola which pass through (x_1, y_1). But these are mutual perpendicular

$$\therefore m_1 \cdot m_2 = -1$$

$$\Rightarrow \frac{y_1^2 + b^2}{x_1^2 - a^2} = -1$$

$\Rightarrow y_1^2 + b^2 = -x_1^2 + a^2$

$\Rightarrow x_1^2 + y_1^2 = a^2 - b^2$

Therefore, required locus is

$x^2 + y^2 = a^2 - b^2$

1. Find the equation of hyperbola whose focus is (2, 2), eccentricity 2 and directrix $x + y = 9$

Let $P(x, y)$ be any point on the hyperbola whose focus is (2, 2) and directrix is $x + y - 9 = 0$

If PM = perpendicular from P on the directrix

then $\dfrac{\text{Distance of } P(x,y) \text{ from } (2,2)}{\text{Distance of } P \text{ from directrix } x+y-9=0} = e$

$$\Rightarrow \frac{\sqrt{(x-2)^2 + (y-2)^2}}{\frac{x+y-9}{\sqrt{2}}} = 2$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-2)^2} = \frac{2(x+y-9)}{\sqrt{2}}$$

On squaring

$$(x-2)^2 + (y-2)^2 = \frac{4(x+y-9)^2}{2}$$

$$\Rightarrow x^2 - 4x + 4 + y^2 - 4y + 4 = 2\left[(x+y)^2 + 81 - 2(x+y) \cdot 9\right]$$

$$\Rightarrow x^2 + y^2 - 4x - 4y + 8 = 2\left[x^2 + 2xy + y^2 + 81 - 18x - 18y\right]$$

$$\Rightarrow x^2 + y^2 - 4x - 4y + 8 = 2x^2 + 4xy + 2y^2 + 162 - 36x - 36y$$

$$\Rightarrow x^2 + y^2 + 4xy - 32x - 32y + 154 = 0$$

Find the eccentricity, the co-ordinate of the foci, equation of the directrices, the axes and latus rectum.

When the equation of hyperbola is

Solution :→ $4x^2 - 9y^2 = 36$

the equation of hyperbola is

$$4x^2 - 9y^2 = 36$$

$$\Rightarrow \frac{4x^2}{36} - \frac{9y^2}{36} = 1$$

$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1 \quad \text{— (1)}$$

Compare it with the standard equation of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\Rightarrow a^2 = 9 \Rightarrow a = 3$

$b^2 = 4 = a^2(e^2-1)$

$\Rightarrow 4 = 9(e^2-1)$

$\Rightarrow \frac{4}{9} = e^2-1$

$\Rightarrow e^2 = \frac{4}{9}+1 = \frac{13}{9}$

$\Rightarrow e = \frac{\sqrt{13}}{3}$ which is required eccentricity

Equation of directrices $\Rightarrow x = \pm a/e$

$= \pm \frac{3\times 3}{\sqrt{13}} = \pm\frac{9}{\sqrt{13}}$

Equation of axes $y = 0$

$x = 0$

foci are $(\pm ae, 0)$ Now $ae = \pm \frac{3\times\sqrt{13}}{3} = \pm\sqrt{13}$

∴ foci are $(\pm\sqrt{13}, 0)$

length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 4}{3} = 8/3$.